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Odd Zonal Harmonics in the Geopotential, Determined from Fourteen Well-Distributed Satellite Orbits

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Odd zonal harmonics in the geopotential, determined from fourteen well-distributed satellite orbits

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Coefficients of the odd zonal harmonics in the Earth's gravitational potential are evaluated by analysing the oscillations in orbital eccentricity of fourteen satellites chosen to give the widest and most uniform possible distribution in orbital inclination and semi-major axis. The best representations of the odd zonal harmonics are found to be in terms of seven coefficients (J_3, J_5, ..., J_{15}) or ten coefficients (J_3, J_5, ..., J_{21}) and values for these coefficients are given.

A detailed account of this work is being published in *Planetary and Space Science*.

1. Introduction

In this paper we are not concerned with the variations of gravitational potential with longitude, but only with the zonal harmonics, which indicate the variations with latitude, averaged over all longitudes. Furthermore, we are studying only the odd zonal harmonics, which specify that part of the gravitational field which is not symmetrical about the equator. We write the gravitational potential U at an exterior point distant r from the Earth's centre, and having geocentric latitude ϕ, as a series of spherical harmonics

$$U = \frac{GM}{r}\left\{1 - \sum_{n=2}^{\infty} J_n \left(\frac{R}{r}\right)^n P_n(\sin\phi)\right\}, \tag{1}$$

where G is the gravitational constant, M the mass of the Earth, and R the Earth's equatorial radius. P_n (sin ϕ) is the Legendre polynomial of degree n and argument sin ϕ, and the J_n are constant coefficients. GM is taken as 398 602 km^3/s^2 and R as 6378·163 km.

Our aim is to determine as many as possible of the odd harmonics in equation (1): we attempt to evaluate J_3, J_5, J_7, ..., J_{2m-1}, where m is taken as large as seems practicable, on the assumption that J_{2m+1}, J_{2m+3}, ... are zero.

2. Method

The coefficients of the odd zonal harmonics can best be determined from the amplitudes of the long-period oscillations which they cause in the orbital elements of close Earth satellites and in particular from the oscillations in orbital eccentricity. If β is the amplitude of the observed oscillation in eccentricity we obtain an equation for the coefficients of the odd harmonics J_3, J_5, J_7, ... of the form

$$A_3 J_3 + A_5 J_5 + A_7 J_7 + \ldots = \beta,$$

where A_3, A_5, A_7 are constant for a particular orbit and depend mainly on the orbital inclination i and the semi-major axis a. Several equations of this type can be obtained by using several different satellites, and these simultaneous equations can then be solved if all harmonics beyond a certain degree are ignored. The more satellites we have, and the more evenly distributed their orbits are, the better the determination is likely to be. For our new determination we tried to obtain the widest possible distribution in inclination and semi-major axis, and after examining all the orbits available, we chose the seventeen orbits indicated in figure 1. This shows that we have quite a good coverage in inclination and semi-major axis but that we lack any satellite of inclination less than 28° and we have no satellite with a large semi-major axis, and with an inclination between 50° and 80°. The coverage is, however, much better than in any previous determination of the zonal harmonics.

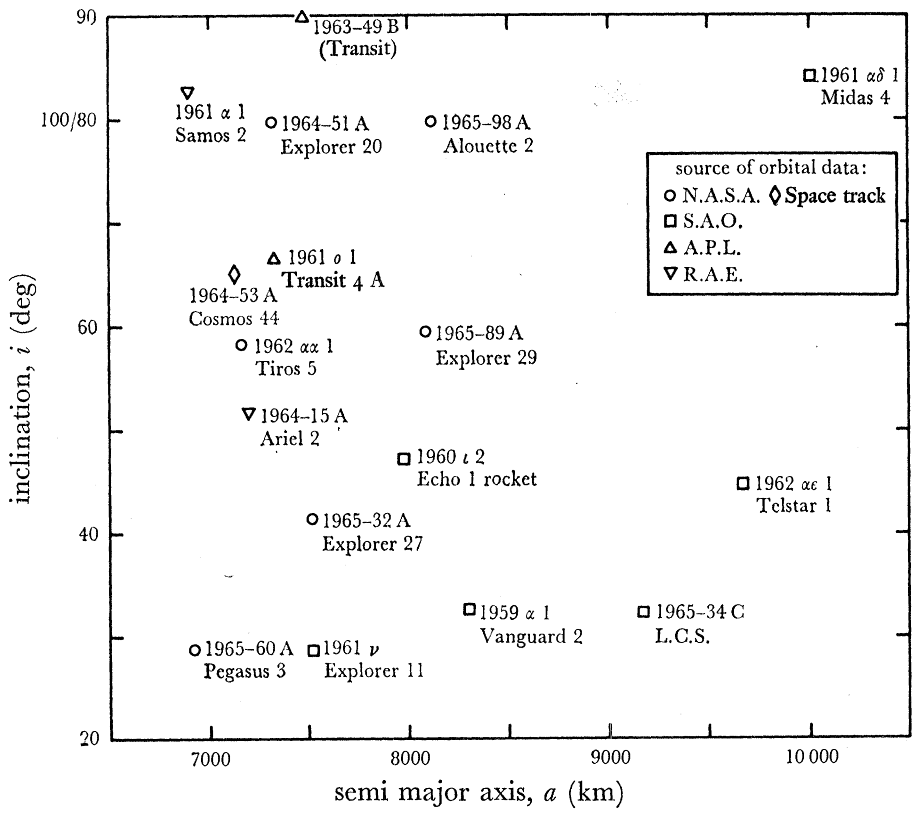

FIGURE 1. Inclinations and semi major axes of satellite orbits chosen for analysis.

The detailed procedure is generally the same as in our previous papers (King-Hele & Cook 1965; King-Hele, Cook & Scott 1965). We find the amplitude β of the oscillation in eccentricity after removal of perturbations due to drag, luni-solar effects, and the even harmonics. If the eccentricity is not too small the variation should theoretically be sinusoidal and its amplitude is found by fitting a sine curve to the observed points. For small-eccentricity orbits, however, the variation of eccentricity with argument of perigee departs significantly from a sine curve. The most convenient way of analysing these orbits (Cook 1966) is to plot $\xi = e \cos \omega$ against $\eta = e \sin \omega$. The points should lie on a circle, with the radius of the circle usually giving the mean eccentricity and the centre of the circle being at a distance β from the ξ axis.

3. RESULTS

The seventeen equations which we obtained for the coefficients of the odd harmonics were solved for various numbers of coefficients. It became apparent that two of our satellites, Cosmos 44 at an inclination of 65°, and Samos 2, at an inclination of 97°, were so sensitive to the neglected harmonics, of degree higher than 20, that they were upsetting the solutions obtained on the assumption that these harmonics were zero. We therefore

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decided to drop these two satellites temporarily, and we also dropped Explorer 29, at an inclination of 59°, which did not fit in at all with the other satellites.

This left fourteen equations, which we solved by the least-squares method for 6, 7, 8, 9, 10 and 11 coefficients. The sums of squares of the weighted residuals, Σ, decreased by a factor of 8 when N was increased from 6 to 7 and by a factor of 4 when N was increased from 9 to 10. This suggests that we must regard the 7-coefficient and 10-coefficient solutions as better than their neighbours.

It is of course rather doubtful whether the 10-coefficient solution from fourteen equations can be regarded as legitimate, but by a stroke of good fortune two of the coefficients of our 10-coefficient solution were zero, namely J_9 and J_{19}. We therefore solved the equations again for 8 coefficients on the assumption that J_9 and J_{19} were zero. The values obtained were the same as in the 10-coefficient solution, but their standard deviations were reduced. This solution carried us up to J_{21}. Further solutions including J_{23} and J_{25} suggested that the values of these two higher coefficients were small and they were therefore not worth taking into account.

So we present as our preferred sets of values the 7-coefficient solution, and the 8-coefficient solution with two additional zero coefficients, which we will refer to as our 10-coefficient solution. These values are given in table 1.

TABLE 1. SETS OF VALUES FOR THE ODD HARMONIC COEFFICIENTS

	7-coefficient	10-coefficient
$10^6 J_3$	$-2{\cdot}53 \pm 0{\cdot}02$	$-2{\cdot}50 \pm 0{\cdot}01$
$10^6 J_5$	$-0{\cdot}22 \pm 0{\cdot}04$	$-0{\cdot}26 \pm 0{\cdot}01$
$10^6 J_7$	$-0{\cdot}41 \pm 0{\cdot}06$	$-0{\cdot}40 \pm 0{\cdot}02$
$10^6 J_9$	$+0{\cdot}09 \pm 0{\cdot}06$	$0 \pm 0{\cdot}06$
$10^6 J_{11}$	$-0{\cdot}14 \pm 0{\cdot}05$	$-0{\cdot}27 \pm 0{\cdot}06$
$10^6 J_{13}$	$+0{\cdot}29 \pm 0{\cdot}06$	$+0{\cdot}36 \pm 0{\cdot}08$
$10^6 J_{15}$	$-0{\cdot}40 \pm 0{\cdot}06$	$-0{\cdot}65 \pm 0{\cdot}10$
$10^6 J_{17}$	—	$+0{\cdot}30 \pm 0{\cdot}08$
$10^6 J_{19}$	—	$0 \pm 0{\cdot}11$
$10^6 J_{21}$	—	$+0{\cdot}58 \pm 0{\cdot}11$
Σ	3·41	0·79

4. DISCUSSION

4·1. *Comparison of the 7- and 10-coefficient solutions*

First it should be emphasized that the standard deviations quoted here in, for example, the 7-coefficient solution give an indication of the errors when the potential is represented by 7 coefficients, but are not necessarily consistent with the values in the 10-coefficient solution. In particular, the effect of the large neglected J_{21} term will be 'shared out' among the other coefficients in the 7-coefficient solution. To allow for the neglect of higher-degree harmonics it would perhaps be wise to assume that the standard deviations in the last three coefficients in both sets should be increased, perhaps by a factor of between 1·5 and 2, if they are to provide realistic error estimates. It can be argued that all the standard deviations in the 10-coefficient solution should be increased by a factor of 2 because the fit is so good.

There is a danger that the series of coefficients J_3, J_5, J_7, ... will develop an oscillatory tendency with alternating large positive and negative values, if too many coefficients are being evaluated. In fact there was little sign of such a tendency even in the 11-coefficient

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solution which we obtained. However, it may be present in a mild degree, and in fact there is some sign of it in the 7-coefficient solution, though this may not be significant.

If a choice has to be made between the 7- and 10-coefficient solutions we should logically recommend the 10-coefficient solution because it has such a low value of Σ, and because the 7-coefficient solution neglects J_{21}. However, if caution were allowed to override logic, the 7-coefficient solution might be recommended because comparison with the 10-coefficient solution shows quite good agreement, and there is no such check possible on the 10-coefficient solution.

4·2. *Comparison with previous results*

No previous sets of values go beyond J_{13}, and since J_{15} is large the previous sets cannot be regarded as entirely satisfactory. The three most complete sets of values previously available are compared with the 7-coefficient solution in table 2.

Table 2 indicates that the values of J_3 to J_9 derived by King-Hele, Cook & Scott (1965) and by Kozai (1964) are quite satisfactory, being within two standard deviations of the values in the 7-coefficient set. The values obtained by Guier & Newton (1965) do not satisfy this rather stringent test, and this is not altogether surprising because they were derived from only four satellites. Kozai's value of J_{11} does not agree with ours.

Table 2. Comparison with previous results

	$10^6 J_3$	$10^6 J_5$	$10^6 J_7$	$10^6 J_9$	$10^6 J_{11}$	$10^6 J_{13}$	$10^6 J_{15}$
Guier & Newton (1965)	−2·68	−0·03	−0·59	0·18	—	—	—
King-Hele, Cook & Scott (1965)	−2·56	−0·15	−0·44	0·12	—	—	—
Kozai (1964)	−2·56	−0·18	−0·38	0·04	0·30	—	—
7-coefficient solution	−2·53	−0·22	−0·41	0·09	−0·14	0·29	−0·40
s.d.	2	4	6	6	5	6	6

5. Conclusions

In conclusion it is perhaps worth looking back over the past 7 years to see the progress made in evaluating the odd zonal harmonics. A reasonably good value for the third harmonic, J_3, was found by O'Keefe, Eckels & Squires (1959). Reliable sets of values for J_3, J_5 and J_7 emerged in the years 1961–63 in the papers by Newton, Hopfield & Kline (1961), Kozai (1962) and Smith (1963). Consistent sets of values for J_3, J_5, J_7 and J_9 were obtained in 1964–65 (with an uncertain value for J_{11}) in the papers by Kozai (1964), Guier & Newton (1965), and King-Hele, Cook & Scott (1965). In the present analysis the evaluation has been carried to J_{21} with an indication that J_{23} and J_{25} are small.

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